Progress of Theoretical Physics, Vol. 118, No. 3, September 2007

Ground State Properties with a Random Two-Body Interaction

Noritaka SHIMIZU[1] and Takaharu OTSUKA[1,2,3]

[1] *Department of Physics, The University of Tokyo, Tokyo 113-0033, Japan*
[2] *Center for Nuclear Study, University of Tokyo, Tokyo 113-0033, Japan*
[3] *RIKEN, Wako 351-0198, Japan*

(Received April 19, 2007)

We apply a two-body random ensemble to a hypothetical system of spinless fermions occupying a one-dimensional rotationally invariant orbit in order to study the statistical tendencies of spins of ground states for a random two-body interaction. We also investigate the origin of the dominance of spin-zero ground states of a single-j shell in the nuclear shell model, which can be interpreted geometrically as a particular case of the simple hypothetical model we consider. Assuming a specific symmetry, or the doublet structure due to time-reversal symmetry, it is found that the ensemble has a preference for a ground state whose spin is zero. We demonstrate how the time-reversal symmetry and its doublet structure affect this ground state tendency.

§1. Introduction

Random matrix theory is one of the most useful methods for studying the statistical properties of quantum chaos in many-body systems whose interactions are not clearly understood.[1)] Because many physical objects are assumed to have a Hamiltonian consisting of one-body and two-body interactions, the two-body random ensemble (TBRE) has been used in many studies investigating the statistical properties of the spectra of many-body systems.[2),3)] The one-body interaction is often omitted for simplicity.

Johnson et al.[4)] found that the ground states in the nuclear shell model, obtained with the TBRE, tend to have $J^\pi = 0^+$ in even-even nuclei, even though the size of the $J^\pi = 0^+$ subspace is very small in comparison with that of the entire Hilbert space. The search for the origin of this phenomenon, which is called "$J = 0$ dominance," has been an important topic since that finding. In studies of this topic, the Gaussian orthogonal ensemble (GOE) in two-body system is often used to make the TBRE preserving time-reversal symmetry.[2),3)] In the GOE, the Hamiltonian preserves time-reversal symmetry, and its matrix elements are real numbers generated randomly, according to the Gaussian distribution.

In the case of realistic nuclei, the ground states of even-even nuclei have $J^\pi = 0^+$, without any exception. This phenomenon is attributed to be the pairing correlation. In the random case, Bijker et al. reported that time-reversal symmetry cannot be the origin of the $J = 0$ dominance due to random interactions, by introducing a Gaussian unitary ensemble (GUE) instead of GOE.[5)] In the GUE, each matrix element of the Hamiltonian is taken to be a complex number, and time-reversal symmetry is broken. Johnson et al. also showed that the generalized seniority produced by a random Hamiltonian possesses a robust pairing feature in the two-body interactions.[6)] In addition, Ref. 9) suggests that the two-body states of time-reversal pairs increase the trace of the Hamiltonian matrix of even J states in the TBRE. Various further investigations have been done to search for the origin of the $J = 0$ dominance,[7),8),10)–12)] but it is not yet clearly understood. Spectral widths[10)] and energy centroids[9)] were investigated by focusing on eigenvalue distributions.

Another direction of study has been the use of simplified models. For example, Zhao et al.[13),14)] studied a single $j = 7/2$ shell system, which is so simple that its exact eigenstates can be obtained analytically, and found empirical rules for the probabilities of the ground state spin. Kaplan et al.[15)] studied a system composed of spin 1/2 with zero orbital angular momentum. That system exhibits strong $J = 0$ dominance without any indication of pairing correlation. They conjectured that the $J = 0$ dominance arises through "the coupling of time-reversed fermions". In addition, they studied a spinless system and demonstrated the localization of the eigenstates in Fock space, although the low-lying spectra were not studied.[16)]

In this work, we elucidate the role of time-reversal symmetry and the pair structure of the two-body interaction in the $J = 0$ dominance phenomenon. For this purpose, we introduce a hypothetical one-dimensional system consisting of spinless fermions under periodic conditions, which is geometrically much simpler than the three-dimensional system in the nuclear shell model. Because the study of the $J = 0$ dominance of the nuclear shell model is very complex, due to the preservation of the rotational symmetry of the three-dimensional space, we introduce a two-dimensional rotation with a one-dimensional variable and omit the intrinsic spin of fermions. This system has a good quantum number M, corresponding to the z-component of the angular momentum, J_z, in the case of three-dimensional rotation. One of our aims is to study the ground-state properties of this one-dimensional system.

We describe this hypothetical system in §§2 and 3, and its ground-state properties in detail in §4. For simplicity, we introduce a small one-dimensional system and analytically investigate it in §5. In §6, we study a single-j system in the nuclear shell model for comparison. Finally, we summarize this work in §7.

§2. Spinless fermions with preserved rotational symmetry

We introduce a hypothetical one-dimensional system with spinless fermions under periodic boundary conditions. An orbit of these spinless fermions is taken to be a circle in two-dimensional space. Because this system is assumed to preserve rotational invariance, it has a good quantum number M corresponding to the quantum number of the rotational symmetry.

We define a single-particle wave function of this system as

$$\langle \phi | m \rangle = e^{im\phi}, \tag{2·1}$$

where ϕ is the angle of the position P in Fig. 1, and m is the z-component of the angular momentum. Because this system consists of spinless fermions, it is reasonable that m is regarded as an integer. There are $2m_m + 1$ single-particle states, $|-m_m\rangle, |-m_m+1\rangle, |-m_m+2\rangle, ...|m_m-1\rangle$, and $|m_m\rangle$, where m_m is the largest value of m. In addition, we consider the fictitious case in which m is half integer, in order to facilitate comparison with the case of the single-j shell system in the nuclear shell model considered in §6.

In the second-quantized form, the creation operator for a spinless fermion whose z-component of the angular momentum is m, $c_m^\dagger$, is defined as

$$|m\rangle = c_m^\dagger|-\rangle, \qquad (2\cdot2)$$

where $|-\rangle$ is the vacuum state.

A general two-body interaction of this system is written as

$$\hat{H} = \sum_{m_1\geq m_2, m_3\geq m_4} V_{m_1m_2m_3m_4} c_{m_1}^\dagger c_{m_2}^\dagger c_{m_4} c_{m_3}, \qquad (2\cdot3)$$

where the two-body matrix element $V_{m_1m_2m_3m_4} = V_{\alpha,\alpha'}$ is written as

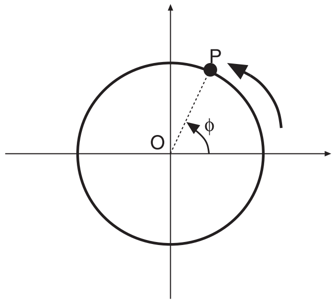

Fig. 1. Schematic picture of the system consisting of spinless fermions with rotational symmetry.

$$V_{m_1m_2m_3m_4} = \langle m_1, m_2|\hat{H}|m_3, m_4\rangle. \qquad (2\cdot4)$$

We next introduce $\alpha \equiv (m_1, m_2)$, which represents a two-body state with $m = m_1$ and m_2, and using it, Eq. (2·4) is rewritten as

$$V_{\alpha,\alpha'} = \langle\alpha|\hat{H}|\alpha'\rangle, \qquad (2\cdot5)$$

where $V_{\alpha,\alpha'}$ is a hermitian matrix.

We assume that the Hamiltonian is invariant under rotation about the origin of the coordinate axes, O, in Fig. 1 throughout this work. The operator of the rotation, $\hat{J}_z$, satisfies

$$\hat{J}_z|m\rangle = m|m\rangle. \qquad (2\cdot6)$$

Because it commutes with the Hamiltonian in Eq. (2·3), the total angular momentum M can be taken as a good quantum number. We apply to the Hamiltonian matrix in Eq. (2·4) the condition

$$V_{m_1m_2m_3m_4} = 0 \quad \text{if} \quad m_1 + m_2 \neq m_3 + m_4, \qquad (2\cdot7)$$

so that the Hamiltonian preserves the rotational symmetry. Hereafter, we refer to this one-dimensional system in which rotational symmetry is preserved on the "M system". We apply the TBRE to the M system and study its ground state properties in the following sections.

§3. Systems with and without time-reversal symmetry

For our time-reversal symmetry preserving ensemble, we consider the two-body matrix elements of the TBRE provided by the GOE,

$$\overline{V_{\alpha,\alpha'}} = 0, \qquad (3\cdot1)$$

$$\overline{V^2_{\alpha,\alpha'}} = v^2(1+\delta_{\alpha\alpha'}) \quad (3\cdot2)$$

and

$$\overline{V_{\alpha,\alpha'}V_{\beta,\beta'}} = 0 \qquad \text{if} \ \ (\alpha,\alpha') \neq (\beta,\beta'), \quad (3\cdot3)$$

where $V_{\alpha,\alpha'}$ is a real number generated randomly according to a Gaussian distribution, v is a constant, and $\overline{V^2}$ denotes the ensemble average of V^2. If time-reversal symmetry is assumed in the Hamiltonian of the nuclear shell model, all the matrix elements can be selected as real numbers, and the GOE is suitable.

In the M system, one more condition is required to preserve time-reversal symmetry. The time reversal operator $\hat{\Theta}$ is defined as

$$\hat{\Theta} = \hat{K}\hat{U}, \quad (3\cdot4)$$

where $\hat{U}$ is a unitary operator and $\hat{K}$ is the complex conjugation operator, which operates only on the coefficients of the wave functions, but not on the wave functions themselves.[17] The basis $|m\rangle$ is transformed by $\hat{U}$ as

$$|-m\rangle = \hat{U}|m\rangle. \quad (3\cdot5)$$

A general wave function, which is written as a linear combination of the basis states $|m\rangle$, is transformed by this operator as

$$\hat{\Theta}\sum_m f_m|m\rangle = \sum_m \hat{K} f_m \hat{U}|m\rangle = \sum_m f^*_m|-m\rangle, \quad (3\cdot6)$$

where f_m is the coefficient of the linear combination. The operator K operates only on the coefficient f_m, not the basis state $|m\rangle$, and therefore it depends the basis and does not have significant physical meaning independently.

The strengths of the two-body interaction are limited to real numbers and satisfy the condition

$$V_{m_1m_2m_3m_4} = V_{-m_2-m_1-m_4-m_3} \quad (3\cdot7)$$

in order to preserve the time-reversal symmetry of the Hamiltonian. The GOE under the condition Eq. (3·7) preserves time-reversal symmetry. Note that Eq. (3·7) represents only invariance under the transformation generated by the operator $\hat{U}$, not time-reversal invariance. The symmetry with respect to the transformation generated by the operator $\hat{U}$ corresponds to the equivalence of clockwise and counterclockwise rotations. We can interpret this symmetry as a geometrical property of the Hamiltonian. This symmetry can also be interpreted as a line symmetry around the x-axis, or reflection symmetry. For the Hamiltonian which is invariant under the $\hat{U}$ transformation, the states $|m\rangle$ and $|-m\rangle$ are necessarily degenerate. Hereafter, we refer to these two degenerate states and the symmetry as the U-doublet and U symmetry, respectively.

In order to investigate the effect of the time-reversal symmetry and its complex conjugate operator, we introduce the GUE, which breaks the time-reversal symmetry. In this ensemble, we consider a two-body Hamiltonian matrix in Eq. (2·3) of the form

$$V_{\alpha\alpha'} = \frac{S_{\alpha\alpha'} + i\epsilon A_{\alpha\alpha'}}{\sqrt{1+\epsilon^2}}, \quad (3\cdot8)$$

where i is the imaginary unit, and $S_{\alpha\alpha'}$ and $A_{\alpha\alpha'}$ are real symmetric and antisymmetric matrices, respectively.[5] The matrix elements of $S_{\alpha\alpha'}$ and $A_{\alpha\alpha'}$ are randomly generated according to Gaussian distributions:

$$\overline{S^2_{\alpha\alpha'}} = v^2(1 + \delta_{\alpha\alpha'}) \tag{3·9}$$

and

$$\overline{A^2_{\alpha\alpha'}} = v^2(1 - \delta_{\alpha\alpha'}). \tag{3·10}$$

In the present work, ϵ is taken to be 1, except where we particularly mention. We construct the GUE so that it preserves the U symmetry but, at the same time, breaks the time-reversal symmetry in Eq. (3·4). We study the ground state properties of the GUE in the next section.

§4. Ground state properties of the M system

In this section, we present some results concerning the ground state properties of the ensembles of M systems consisting of 6 identical particles in the $m_m = 11/2$ orbit and discuss the effect of symmetries possessed by the Hamiltonian.

Figure 2 plots the probability of the ground state with $M = M_{\rm gs}$ for various ensembles, $P(M_{\rm gs})$. The plots in Fig. 2(a) show the probability of the ground states for

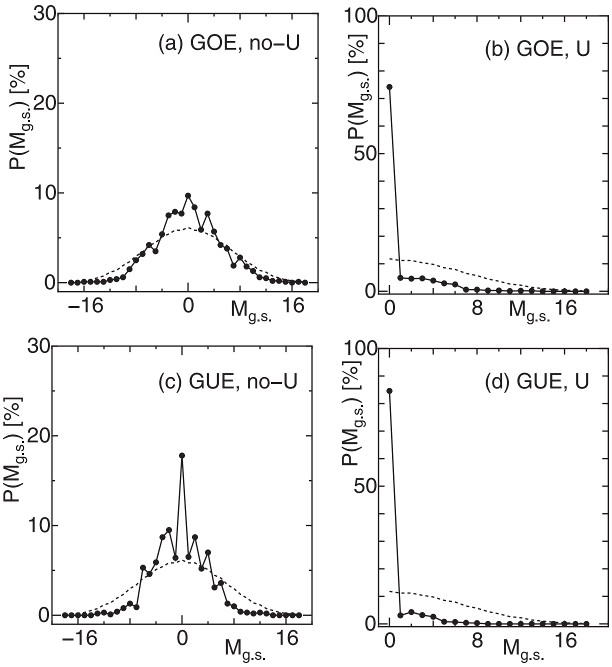

Fig. 2. Probability of the ground state spins for 6 identical particles in the $m_m = 11/2$ system. These results were obtained using (a) GOE without the U symmetry, (b) GOE with the U symmetry, (c) GUE without the U symmetry, and (d) GUE with the U symmetry, respectively. They were obtained from 1000 runs of the TBRE. The dotted curves represent the ratios of the sizes of the Hilbert spaces of $M = M_{\rm gs}$ to that of the entire space.

a range of values of $M_{\rm gs}$ in the GOE of the M system without assuming U symmetry. Although $P(M_{\rm gs}=0)$ is larger than $P(M_{\rm gs})$ for other values of $M_{\rm gs}$, it is not large in comparison with the case in which there exists time-reversal symmetry. We can conclude that $P(M_{\rm gs})$ without time-reversal symmetry is consistently proportional to the ratio of the sizes of the Hilbert spaces of $M = M_{\rm gs}$ to that of the entire space, which is represented by the dotted curve in Fig. 2(a).

The case considered in Fig. 2(b) is the same as that in (a), except for the existence of the U symmetry. In this case, the ensemble preserves time-reversal symmetry. Because the states with the value M and $-M$ are necessarily degenerate, the probabilities for negative values of M are omitted. Surprisingly, $P(M_{\rm gs}=0)$ in Fig. 2(b) is prominently large in comparison with the size of the Hilbert space with $M = 0$. Hereafter, we refer to the property that the ground state tends to have $M = 0$ as "$M = 0$ dominance".

The plots in (c) and (d) are the results of the GUE with and without U-symmetry. It is seen that their behavior is almost the same as that in the cases (a) and (b), respectively. The plot in (c) shows $M = 0$ dominance even though it does not have time-reversal symmetry. Noting this fact, we conjecture that the origin of the $M = 0$ dominance is the symmetry expressed in Eq. (3·7), resulting from the U symmetry, not time-reversal symmetry.

Regarding the nuclear shell model, Bijker et al. showed that the GUE also yields $J = 0$ dominance and that the probability for a $J = 0$ ground state in the GUE is slightly larger than that in the GOE.[5)] In their work, the structure of the two-body interaction they used is preserved in the ensemble, and only the time-reversal symmetry is broken, not the U symmetry, which is realized, e.g., by Eq. (3·7). It seemed that the origin of the $J = 0$ dominance is not time-reversal symmetry but the symmetry expressed by Eq. (3·7), i.e., the U symmetry.

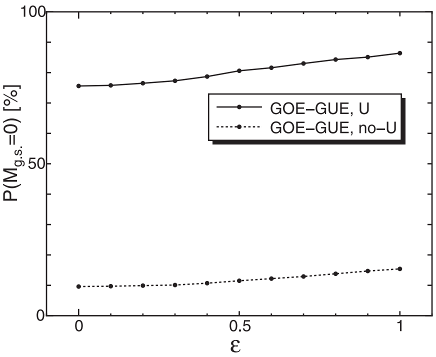

Fig. 3. Probability that the ground state spins satisfy $M_{\rm gs} = 0$ for 6 identical particles in the $m_m = 11/2$ system as a function of the quantity ϵ appearing in Eq. (3·8). The symbols with the solid curve represent the results with U symmetry, and those with the dotted line represent the results without U symmetry. These data were obtained from 1000 runs of the TBRE.

In the present result, we have shown that $M = 0$ dominance occurs in a system with time-reversal symmetry and does not occur in a system without time-reversal symmetry. On the other hand, both systems with time-reversal symmetry and without time-reversal symmetry exhibit $J = 0$ dominance in the three-dimensional nuclear-shell-model system.[5)]

The difference between the present result and Bijker's work regards the question of whether the invariance under the U transformation is preserved in the system without time-reversal symmetry. In that work, the structure of V, or the condition in Eq. (3·7), is pre-

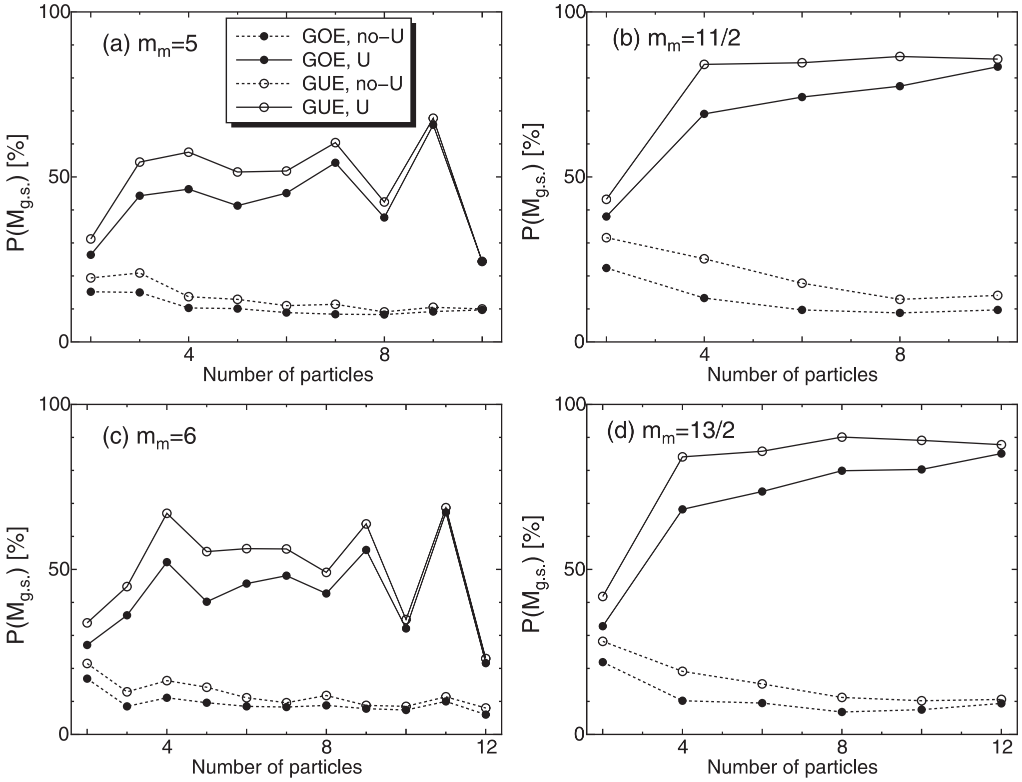

Fig. 4. Probabilities that $M_{\rm gs} = 0$ in the $m_m = 5$ (a), $m_m = 11/2$ (b), $m_m = 6$ (c), and $m_m = 13/2$ (d) systems as a function of the number of particles. These data were obtained from 1000 runs of the TBRE.

served and only the matrix element of the two-body interaction is enhanced to a complex number obeying the GUE. The tendency of the ground-state spin is changed very little. In the present work concerning the M system discussed in §3, the corresponding study is that depicted in Figs. 2(b) and (c), although we can also consider the M system without the condition given in Eq. (3·7).

Figure 3 shows how $P(M_{\rm gs} = 0)$ depends the quantity ϵ appearing in Eq. (3·8). It is seen that $P(M_{\rm gs} = 0)$ smoothly increase as a function of ϵ. This tendency is similar to that in the case of the nuclear shell model.[5)]

Figure 4 plots $P(M_{\rm gs} = 0)$ for systems with $m_m = 5, 11/2, 6,$ and $13/2$. We omit the case of an odd number of particles in the $m_m = 11/2$ and $m_m = 13/2$ systems, because no many-body $M = 0$ state exists. In these figures, the values of $P(M_{\rm gs} = 0)$ for the GUE are slightly larger than those for the GOE, and these probabilities for the ensembles with U symmetry are far larger than those without U symmetry. We have checked many systems with values of m_m smaller than $13/2$, and we found that a similar tendency exists in systems larger than that of 3 particles with $m_m = 2$. The smallest system having this tendency is discussed in §5. In systems with half-integer values of m_m, such as $m_m = 11/2$ and $m_m = 13/2$, we can also see strong staggering behavior in the case of a small number of particles or holes.

When we take the matrix elements to be complex numbers distributed according to the GUE under the condition given in Eq. (3·7), the M system has U symmetry. From the results presented in this section, because $M = 0$ dominance exists only in the case of U symmetry, we can conclude that the origin of the $M = 0$ dominance is the U symmetry possessed by the Hamiltonian.

§5. Simple case of the M system

In this section, we demonstrate how U symmetry brings about $M = 0$ dominance through study of an analytically solvable model. We investigate a system consisting of 3 particles in $m_m = 2$ orbits, which is the smallest system for which $M = 0$ dominance is realized.

Here, we assume that the system is rotationally invariant without U symmetry. The two-body basis in this system is defined as

$$\begin{aligned}
|L=-3\rangle &= c^\dagger_{-1}c^\dagger_{-2}|-\rangle, \\
|L=-2\rangle &= c^\dagger_{0}c^\dagger_{-2}|-\rangle, \\
|L=-1,1\rangle &= c^\dagger_{1}c^\dagger_{-2}|-\rangle, \\
|L=-1,2\rangle &= c^\dagger_{0}c^\dagger_{-1}|-\rangle, \\
|L=0,1\rangle &= c^\dagger_{1}c^\dagger_{-1}|-\rangle, \\
|L=0,2\rangle &= c^\dagger_{2}c^\dagger_{-2}|-\rangle, \\
|L=1,1\rangle &= c^\dagger_{2}c^\dagger_{-1}|-\rangle, \\
|L=1,2\rangle &= c^\dagger_{1}c^\dagger_{0}|-\rangle, \\
|L=2\rangle &= c^\dagger_{2}c^\dagger_{0}|-\rangle, \\
|L=3\rangle &= c^\dagger_{2}c^\dagger_{1}|-\rangle,
\end{aligned} \tag{5·1}$$

where L is the expectation value of the angular momentum $\langle \hat{J}_z \rangle$ of the two-body system. For example, $|L=-1,2\rangle$ denotes the second state of $L=-1$.

The two-body Hamiltonian is rewritten as

$$\begin{aligned}
\hat{H} &= \sum_{m_1 \geq m_2, m_3 \geq m_4} V_{m_1 m_2 m_3 m_4}\ c^\dagger_{m_1} c^\dagger_{m_2} c_{m_4} c_{m_3} \\
&= \sum_{m_1 \geq \frac{L}{2}, m_3 \geq \frac{L}{2}, L} \delta_{m_1+m_2,L} \delta_{m_3+m_4,L}\ V^L_{m_1,m_3} \\
&\qquad\qquad \times c^\dagger_{m_1} c^\dagger_{m_2} c_{m_4} c_{m_3},
\end{aligned} \tag{5·2}$$

where $V^L_{m_1,m_3}$ is the Hamiltonian matrix whose two-body basis has angular momentum L and is generated randomly according to the Gaussian distribution in Eqs. (3·2) and (3·3) for the TBRE.

Table I lists the ground state probabilities with $M_{\rm gs}$ in the system consisting of $m_m = 3$ and 3 spinless particles. In this table, the row labeled "H. dim" contains the dimension of the Hilbert subspace with the corresponding value of M. The following

Table I. Probabilities of the ground state spins ($M_{\rm gs}$) for various ensembles of the system with $m_m = 2$ and 3 particles. These results were obtained from 1000 runs. (See main text for details.)

$M_{\rm gs}$	-3	-2	-1	0	1	2	3
H. dim.	1	1	2	2	2	1	1
GOE, no-U [%]	7.8	6.7	21.8	25.2	23.8	7.6	7.1
GOE, U [%]	-	-	-	54.4	26.9	12.5	6.2
GUE, no-U [%]	5.5	7.6	25.7	23.6	24.5	6.8	6.3
GUE, U [%]	-	-	-	54.0	30.6	10.6	4.8

rows contain the results for $P(M_{gs})$ in the case of the GOE without U symmetry (GOE, no-U), the GOE with U symmetry (GOE, U), the GUE without U symmetry (GUE, no-U), and the GUE with U symmetry (GUE, U), respectively.

Because the states with M and $-M$ are necessarily degenerate, the probabilities for negative $M_{\rm gs}$ are omitted in the table for the systems with U symmetry. We can see that the probability distribution seems to depend not on the type of ensemble, but on the existence of U symmetry. In the cases with U symmetry there is $M = 0$ dominance, which means a large probability that $M_{\rm gs} = 0$. We thus see that it is the U symmetry which plays an important role in the $M = 0$ dominance, not time-reversal symmetry.

The Hamiltonian matrix in the two-body system is written

$$\begin{aligned}
H^{L=-3} &= (G^{-3}),\\
H^{L=-2} &= (G^{-2}),\\
H^{L=-1} &= \begin{pmatrix} G_1^{-1} & F^{-1} \\ F^{-1*} & G_2^{-1} \end{pmatrix},\\
H^{L=0} &= \begin{pmatrix} G_1^{0} & F^{0} \\ F^{0*} & G_2^{0} \end{pmatrix},\\
H^{L=1} &= \begin{pmatrix} G_1^{1} & F^{1} \\ F^{1*} & G_2^{1} \end{pmatrix},\\
H^{L=2} &= (G^{2}),\\
H^{L=3} &= (G^{3}),
\end{aligned} \tag{5·3}$$

where G and F denote the diagonal and non-diagonal matrix elements, respectively, and the index "$*$" represents complex conjugation. For example, G_1^0 represents the diagonal matrix element whose basis state is $|L = 0, 1\rangle$ in Eq. (5·1), and it is equal to $V^L_{m_1=1,m_1=1}$ in Eq. (2·4).

The basis functions of the three-body system are defined as

$$\begin{aligned}
|M = -3\rangle &= c_0^\dagger c_{-1}^\dagger c_{-2}^\dagger |-\rangle,\\
|M = -2\rangle &= c_1^\dagger c_{-1}^\dagger c_{-2}^\dagger |-\rangle,\\
|M = -1, 1\rangle &= c_2^\dagger c_{-1}^\dagger c_{-2}^\dagger |-\rangle,\\
|M = -1, 2\rangle &= c_1^\dagger c_0^\dagger c_{-2}^\dagger |-\rangle,
\end{aligned}$$

$$\begin{aligned}
|M=0,1\rangle &= c_2^\dagger c_0^\dagger c_{-2}^\dagger|-\rangle,\\
|M=0,2\rangle &= c_1^\dagger c_0^\dagger c_{-1}^\dagger|-\rangle,\\
|M=1,1\rangle &= c_2^\dagger c_1^\dagger c_{-2}^\dagger|-\rangle,\\
|M=1,2\rangle &= c_2^\dagger c_0^\dagger c_{-1}^\dagger|-\rangle,\\
|M=2\rangle &= c_2^\dagger c_1^\dagger c_{-1}^\dagger|-\rangle,\\
|M=3\rangle &= c_2^\dagger c_1^\dagger c_0^\dagger|-\rangle,
\end{aligned} \tag{5·4}$$

where M is the angular momentum of each basis state.

The Hamiltonian matrix elements in this three-body basis are given by

$$\begin{aligned}
H^{M=-3} &= (G_2^{-1} + G^{-2} + G^{-3}),\\
H^{M=-2} &= (G_2^{0} + G_1^{-1} + G^{-3}),\\
H^{M=-1} &= \begin{pmatrix} G_1^1 + G_1^0 + G^{-3} & F^1 \\ F^{1*} & G_2^1 + G_1^{-1} + G^{-2} \end{pmatrix},\\
H^{M=0} &= \begin{pmatrix} G^2 + G_1^0 + G^{-2} & F^0 \\ F^{0*} & G_2^1 + G_2^0 + G_2^{-1} \end{pmatrix},\\
H^{M=1} &= \begin{pmatrix} G^3 + G_1^0 + G_1^{-1} & F^{-1} \\ F^{-1*} & G^2 + G_1^1 + G_2^{-1} \end{pmatrix},\\
H^{M=2} &= (G^3 + G_1^1 + G_2^0),\\
H^{M=3} &= (G^3 + G^2 + G_2^1).
\end{aligned} \tag{5·5}$$

This system does not possess the $M=0$ dominance property, because the distributions of the eigenvalues of the matrix elements $H^{M=-1}$, $H^{M=0}$, and $H^{M=1}$ are all the same.

Now, we apply the condition given in Eq. (3·7) to this system. This condition can be rewritten as

$$\begin{aligned}
G^{-3} &= G^3,\\
G^{-2} &= G^2,\\
G_1^{-1} &= G_1^1,\\
G_2^{-1} &= G_2^1,\\
F^{-1} &= F^1.
\end{aligned} \tag{5·6}$$

From these equations, $H^{M=0}$ and $H^{M=1}$ are formed to be

$$\begin{aligned}
H^{M=0} &= \begin{pmatrix} 2G^2 + G_1^0 & F^0 \\ F^{0*} & 2G_2^1 + G_2^0 \end{pmatrix},\\
H^{M=1} &= \begin{pmatrix} G^3 + G_1^1 + G_1^0 & F^1 \\ F^{1*} & G^2 + G_2^1 + G_1^1 \end{pmatrix}.
\end{aligned} \tag{5·7}$$

Note again that G and F are generated randomly according to the Gaussian distribution, and their variances are $2v^2$ and v^2, respectively. The matrix element $H^{M=-1}$ has the same structure as $H^{M=1}$, due to the degeneracy caused by the U symmetry.

The diagonal matrix elements of $H^{M=1}$ (e.g., $G^3 + G_1^1 + G_1^0$) have variance $6v^2$, while the diagonal matrix elements of $H^{M=0}$ (e.g., $2G^2 + G_1^0$) have larger variance, $10v^2$. The sum of the time-reversal pair of two-body matrix elements (e.g., $2G^2 = G^2 + G^{-2}$) increases the variance of the matrix elements of $H^{M=0}$. Such an increase of the variances occurs only in the $M = 0$ subspace. As a result, the distribution of the eigenvalues of the Hamiltonian with $M = 0$ is broader than those of the other eigenvalues. The degeneracy of the time-reversal pair of the two-body interaction increases the variance of the eigenvalue distribution of $H^{M=0}$. This U doublet could correspond to "the coupling of time-reversed fermions" referred to in Ref. 15).

Thus, the diagonal matrix elements of the Hamiltonian matrix with $M = 0$ are expected to have a relatively broad distribution, due to the degeneracy of the time-reversal-partner state. Generally, the probability of the ground state with M increases with the width of the distribution of H^M. This tendency is independent of whether the ensemble is the GOE or the GUE, and it is consistent with the discussion given in §4.

We thus see that such differences among the variances cause the $M = 0$ dominance, because the probability that a state in a given subspace will be realized as the ground state increases with the relative size of the variance of the eigenvalue distribution of that subspace. In addition, correlation enhances the $M = 0$ dominance. In the nuclear shell model, investigations of the relation between the spectral width of the eigenvalue distribution and the probability of the ground state have been carried out by Papenbrock and Weidenmüller in Refs. 10) and 11).

§6. The nuclear shell-model Hamiltonian in the TBRE

In this section, we study the $J = 0$ dominance in the nuclear shell model and compare it with the $M = 0$ dominance in the M system. In the previous sections, we have elucidated the ground-state properties of a hypothetical one-dimensional system (M system) and demonstrated that the pair structure of the U symmetry in Eq. (3·7) causes $M = 0$ dominance. Because the Hamiltonian in the nuclear shell model is invariant under three-dimensional rotations, it possesses a larger set of symmetries, or more restrictions on the matrix elements of the two-body interaction than in the case of the M system. Our investigation of the $M = 0$ dominance helps us understand the mechanism of the $J = 0$ dominance.

In the nuclear shell model, the Hamiltonian is expected to have rotational symmetry in the three-dimensional space. Because the effect of the one-body interaction is limited, the one-body interaction is omitted in the random matrix theory, whereas a realistic Hamiltonian in the nuclear shell model usually consists of both one-body interactions and two-body interactions. We restrict the model space to a single-j shell for simplicity.

For comparison of this single-j shell and the M system, we regard the z-component of the angular momentum in the single-j shell as the angular momentum of the M system and use the same notation, m_i, as in the M system. Now, we can show that

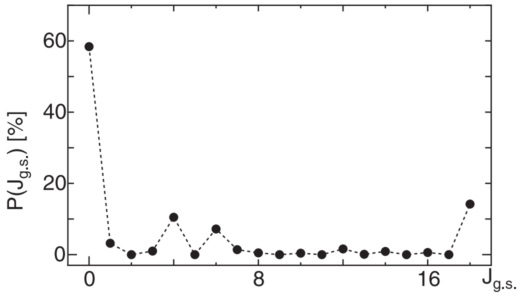

Fig. 5. Probabilities of the ground state spins for 6 identical particles in the single $j = 11/2$ shell. These results were obtained from 1000 runs of the TBRE.

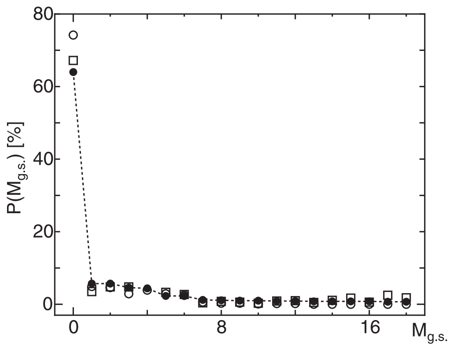

Fig. 6. Probability of the ground state spin. The filled symbols with dotted curve represents the probability for 6 identical particles in the single $j = 11/2$ shell as a function of the z-component of the angular momentum of the ground state, $M_{\rm gs}$. The open boxes represent the data obtained from the TBRE Hamiltonian appearing in Eq. (6·2). The open circles show the ground state probabilities of the M system. They were obtained from 1000 runs using the TBRE Hamiltonian.

the Hamiltonian of the single-j shell satisfies the condition

$$V_{m_1m_2m_3m_4} = \sum_J 2G_J\langle j, m_1, j, m_2|j, m_1+m_2\rangle \times\langle j, m_3, j, m_4|j, m_3+m_4\rangle, \tag{6·1}$$

in Eq. (2·3) using the Clebsch-Gordan coefficients. In the TBRE, G_J is a random number of variance $2v^2$. Thus, we can regard the single-j system as a special case of the M system which satisfies Eq. (6·1).

Figure 5 plots the probability distribution for the spin of the ground states ($J_{\rm gs}$) obtained from the TBRE Hamiltonian of 6 identical particles in the single $j = 11/2$ shell. We can see that the probability of realizing $J_{\rm gs} = 0$ is extremely high, and $J = 0$ dominance exists. At the same time, the probability that $J_{\rm gs}$ is maximal, i.e. 17, is also rather high. This feature is a peculiar characteristic of a single-j shell.[8)]

The form of $V_{m_1m_2m_3m_4}$ in Eq. (6·1) for the nuclear shell-model Hamiltonian satisfies the properties of time-reversal symmetry, given explicitly in Eq. (3·7), according to the symmetry properties of the Clebsch-Gordan coefficients.

In a three-dimensional system, the eigenstates whose total angular momentum is J consist of $2J+1$ degenerate states, whose z-components are $-J, -J+1, -J+2, ..., J-1$ and J. In the M system, three-dimensional rotational symmetry is broken but the one-dimensional rotational symmetry is preserved. Thus, $2J+1$ degeneracy disappears, and only the degeneracy of states having M and $-M$ remains and corresponds to the U-doublet. In order to identify the states, we use only positive values of M for simplicity. Thus, there are $J+1$ states with total angular momentum

J. The z-components of the angular momentum for these states are $0, 1, 2, ..., J-1$ and J. The probability that the ground state spin is M is assumed to be equally distributed by the probability for the ground state spin having J.

The filled circles in Fig. 6 represent the probabilities for the ground state to be spin M, employing the assumption that the probability of the state with spin J is distributed homogeneously over the $M = 0, 1, 2, ..., J$ values of the z-component. We see that the probability that $M = 0$ is significantly larger than the probabilities for the other values.

In order to compare the result for the three-dimensional system with that for the M system, we add small fluctuations that break the three-dimensional rotational symmetry of the original Hamiltonian, such as

$$\hat{H} = \hat{H}_{\mathrm{SM}} + \epsilon \times \hat{H}_{\mathrm{M}}, \tag{6·2}$$

where $\hat{H}_{\mathrm{SM}}$ is the nuclear-shell-model Hamiltonian for a single-j shell, $\hat{H}_M$ is the Hamiltonian of the M system with $m_m = j$, and ϵ is an infinitesimal quantity. The open boxes in Fig. 6 represent the probability for this system with small fluctuations. We see that these results agree quite well with the results obtained using the equal distribution assumption.

In addition, we present the results for the M system assuming time-reversal symmetry, which was discussed in §4. The filled circles represent the probabilities that the ground states have the corresponding values of M. These results are similar to the previous ones obtained using the nuclear-shell-model Hamiltonian. The probability of realizing $M = 0$ is dominantly large, and the probability decreases as a function of M.

The large value of the probability of the $M = 0$ ground state in the M system helps us understand the origin of the $J = 0$ dominance in the nuclear shell model. This observation is consistent with the results obtained in Ref. 9). Note that the nuclear-shell-model Hamiltonian satisfies Eq. (3·7) only for a single-j shell. In many-j shells, another mechanism might produce such $J = 0$ dominance (see, e.g. Ref. 18)).

In Ref. 19), it is shown that the probabilities of $J = 0$ ground states for a single-j system are staggered when $j = 3k - \frac{3}{2}$ (where k is a positive integer). Because this staggering coincides with an increase in the number of $J = 0$ states, we conjecture that it is closely related to the three-dimensional rotational invariance. We cannot see such staggering in the M system.

§7. Concluding remarks

We have investigated the properties of the M system consisting of spinless fermions, and compared this system to the single-j shell system in the nuclear shell model. We also showed how time-reversal symmetry and its geometrical part (U symmetry) affect the $J = 0$ dominance and $M = 0$ dominance phenomena. In the M system, we investigated four ensembles: the GOE without U symmetry, the GOE with U symmetry, the GUE without U symmetry, and the GUE with U symmetry. It was shown that $M = 0$ dominance exists in both of the ensembles with U symmetry.

We demonstrated how U symmetry produces $M = 0$ dominance by studying a small system, both in terms of size and the number of particles, in §5. We found that under U symmetry, the matrix element G^L has the same value as its time-reversal partner matrix element, G^{-L}. If a sufficient number of particles exist and G^L and G^{-L} have very large negative values, these large time-reversal matrix elements tend to form two-particle pairs having L and $-L$. This effect increases the variance of the Hamiltonian matrix elements of $H^{M=0}$, and the ground state of the whole system thus tends to be $M = 0$.

We also investigated the $J = 0$ dominance of the single-j shell system in the nuclear shell model, and we found that the distribution of M for the ground states is similar to that of the M system in the case with U symmetry. The nuclear shell model Hamiltonian also has U symmetry in its structure, and the distribution of the spin of the ground state possesses features similar to the $M = 0$ dominance. It appears that $J = 0$ dominance is closely related to $M = 0$ dominance and U symmetry. The important role of time-reversal pair in the nuclear shell model was also pointed out in Ref. 9).

The U symmetry can be interpreted as a reflection symmetry. Generally, reflection symmetry is considered to be a part of parity symmetry. Studies of ground state properties related to other symmetries, including parity symmetry, in three-dimensional rotationally-invariant systems are in progress.

Acknowledgements

This work was supported mainly by a Grant-in-Aid for Specially Promoted Research (13002001) from MEXT, and by the RIKEN-CNS collaboration project for large-scale nuclear structure calculations. We would like to thank Prof. Bruce R. Barrett for helpful comments and carefully reading the manuscript. One of us (N.S.) wishes to thank Prof. Y. M. Zhao for useful discussions.

References

1) E. P. Wigner, Ann. Math **67** (1958), 325.
2) J. B. French and S. S. M. Wong, Phys. Lett. B **33** (1970), 449.
3) J. B. French and S. S. M. Wong, Phys. Lett. B **35** (1971), 5.
4) C. W. Johnson, G. F. Bertsch and D. J. Dean, Phys. Rev. Lett. **80** (1998), 2749.
5) R. Bijker, A. Frank and S. Pittel, Phys. Rev. C **60** (1999), 021302.
6) C. W. Johnson, G. F. Bertsch, D. J. Dean and I. Talmi, Phys. Rev. C **61** (1999), 014311.
7) Y. M. Zhao, A. Arima and N. Yoshinaga, Phys. Rev. C **66** (2002), 064322.
8) Y. M. Zhao, A. Arima and N. Yoshinaga, Phys. Rev. C **66** (2002), 064323.
9) V. Velázquez and A. P. Zuker, Phys. Rev. Lett. **88** (2002), 072502.
10) T. Papenbrock and H. A. Weidenmüller, Phys. Rev. Lett. **93** (2004), 132503.
11) T. Papenbrock and H. A. Weidenmüller, Phys. Rev. C **73** (2006), 014311.
12) N. Yoshinaga, A. Arima and Y. M. Zhao, Phys. Rev. C **73** (2006), 017303.
13) Y. M. Zhao and A. Arima, Phys. Rev. C **64** (2001), 041301.
14) Y. M. Zhao, A. Arima and N. Yoshinaga, Phys. Rev. C **66** (2002), 034302.
15) L. Kaplan, T. Papenbrock and C. W. Johnson, Phys. Rev. C **63** (2001), 014307.
16) L. Kaplan and T. Papenbrock, Phys. Rev. Lett. **84** (2000), 4553.
17) J. J. Sakurai, *Modern Quantum Mechanics* (Addison-Wesley, 1994).
18) T. Otsuka and N. Shimizu, AIP Conf. Proc. **777** (2005), 135.
19) Y. M. Zhao, A. Arima and N. Yoshinaga, Phys. Rep. **400** (2004), 1.